LIST OF PROBLEMS FOR HAWAII WORKSHOP

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We take for granted the definition of WLP and SLP and do not repeat them here. Unless specified otherwise, all problems assume that the field has characteristic zero and that the algebras in question are graded and artinian. Consequently, the terms "height," "codimension" and "number of variables" will be used interchangeably without comment. The paper [9] has a good overview of known results and open problems concerning the WLP and the SLP, as they relate to the problems in this list. In particular, when we say here that "many authors have studied" a certain problem, a complete list of the authors that we are aware of can be found in [9] and is not repeated here.

1. **Complete Intersections.** It is known that in two or fewer variables, all graded algebras have the WLP [5]. Of course this includes height two complete intersections. In three variables, all complete intersections have the WLP [5]. In any number of variables, any monomial complete intersection has the WLP, and even the SLP. (This was originally shown in [11] and in [13] but has been re-proven in some more recent papers.)

   Is it true that every complete intersection, in any number of variables, has the WLP? Is it true that every height three complete intersection has the SLP? Is it true that every complete intersection in any number of variables has the SLP?

   We believe that the answer will be "yes", at least for the WLP in any number of variables. However, the proof given in [5] for height three does not seem to extend to an arbitrary number of variables.

2. **Gorenstein algebras.** A second very natural extension of the quoted result from [5] would be to show that all codimension three Gorenstein algebras have the WLP. We believe that this is true, but a proof has been very elusive. Some partial results can be found in [10]. Even less is known about the SLP in this setting. It is known that not all codimension four Gorenstein algebras have the WLP. Even among Gorenstein algebras with unimodal Hilbert functions (which is a consequence of the WLP), the WLP does not necessarily hold. For instance, an example in codimension 4 was given by Ikeda [6] in 1996.

3. **Monomial algebras.** It is asking too much to give a complete classification of the monomial algebras that have the WLP. Two natural subclasses come from looking at the end and at the beginning of a minimal free resolution.

   When the summands of the last free module all have the same twist, the algebra is said to be *level*, and the rank of the last free module is called the *type*. The result of [11] and [13] shows that type 1 always has the WLP. It was shown in [1] that level algebras of type 2 in three variables always have the WLP, while every other combination of (≥ 3) variables and type does not necessarily have the WLP. But it may be that given the number of variables and the type, there is an interesting bound on the smallest twist of the summands of the last free module for which the WLP may fail for monomial level algebras. Even less is known about the SLP.

At the other end of the minimal free resolution, one can look for the next case after complete intersections and consider monomial almost complete intersections. Here, several authors have studied what can happen, especially in the case of three variables. What happens when you allow one more generator? What happens when you remove "monomial"?

In the setting of monomial algebras, the results which are known in characteristic zero change dramatically in characteristic p. It often happens that the failure of the WLP can be detected from the vanishing of a certain determinant. In characteristic zero, it is just a question of seeing when the determinant is zero as an integer, but in characteristic p it is a question of determining which primes divide that integer. Again, work of several authors has studied this question, and many interesting connections to combinatorics have been found. But many things remain open. A broad question is the following: given a monomial ideal I such that R/I has the WLP in characteristic zero, what are the field characteristics in which R/I fails to have the WLP? It was shown by Cook and Nagel [2] that this list is always finite, as suggested above.

4. **Powers of linear forms.** Many other natural algebras lend themselves to questions about the WLP. One that we have found interesting is the case when the generators all are powers of linear forms. Again, many authors have studied this problem, and different sorts of results have been found. All of them use a result of Emsalem and Iarrobino [4], which translates the problem to one of studying sets of fat points in projective space. It is beautiful to see how such an algebraic question translates to such a geometric setting. Less is known about the SLP in this setting.

   Here is a specific open problem from [8], Conjecture 6.6.

   > *Let $R = k[x_1, \dots, x_{2n+1}]$, where $n \geq 4$. Let $L \in R$ be a general linear form, and let $I = \langle x_1^d, \dots, x_{2n+1}^d, L^d \rangle$. Then the ring R/I fails the WLP if and only if $d > 1$. Furthermore, if $n = 3$ then R/I fails the WLP when $d = 3$.*

   In the case $n = 3$, $d \geq 4$, it is shown in [8] that the WLP fails, and when $n = 3$, $d = 2$, it was shown that the WLP holds.

5. **Reduced sets of points.** It is known that not every artinian ideal is an artinian reduction of the ideal of a reduced set of points, and certainly is not a *general* artinian reduction of the ideal of a reduced set of points. (This latter condition is deliberately imprecise, just to give a flavor of the problem without bogging down in technicalities.) We would be very interested to know if a general artinian reduction of a reduced, arithmetically Gorenstein set of points has the WLP. If so, the Hilbert functions of such sets of points are completely classified. There are examples of reduced arithmetically Gorenstein sets of points where a *special* artinian reduction fails the WLP, so "general" is important here. Notice that a positive solution to this problem would imply the algebraic g-conjecture (see Murai's list).

6. **Varieties satisfying Laplace equations** A monomial artinian ideal $I \subset R := k[x_0, x_1, \cdots, x_n]$ generated by r forms $F_1, \cdots, F_r$ of degree d is said to be a *monomial Togliatti system* if R/I fails the WLP in degree $d-1$. Example: If $n = 2$ and $d = 3$, $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2)$ is a Togliatti system. The classification of monomial Togliatti systems of cubics has been achieved only when $n = 2$ (see [3] and [12]) and $n = 3$ (see [7]) and its interest relies on the relationship between monomial

Togliatti's systems generated by r forms of degree d in $k[x_0, x_1, \cdots, x_n]$ and n-dimensional varieties satisfying at least one Laplace equation of order $d-1$ (see [7] for more information). We would be interested in the classification of monomial Togliatti systems of cubics.

References

[1] M. Boij, J. Migliore, R. M. Miró-Roig, U. Nagel and F. Zanello, "The shape of a pure O-sequence," Mem. Amer. Math. Soc. **218**, no. 2024 (2012).

[2] D. Cook II and U. Nagel, *Enumerations deciding the Weak Lefschetz Property*, preprint 2011.

[3] D. Franco and G. Ilardi, *On a theorem of Togliatti*, Int. Math. J. **2** (2002), 379–397.

[4] J. Emsalem and A. Iarrobino, *Inverse system of a symbolic power I*, J. Algebra **174** (1995), 1080-1090.

[5] T. Harima, J. Migliore, U. Nagel, and J. Watanabe, *The Weak and Strong Lefschetz Properties for artinian K-Algebras*, J. Algebra **262** (2003), 99–126.

[6] H. Ikeda, *Results on Dilworth and Rees numbes of artinian local rings*, Japan. J. Math. **22** (1996), 147–158.

[7] E. Mezzetti, R.M.Miró-Roig and G. Ottaviani, *Laplace Equations and the Weak Lefschetz Property*, Canadian J. of Math., to appear.

[8] J. Migliore, R. Miró-Roig and U. Nagel, *On the Weak Lefschetz Property for powers of linear forms*, to appear in Algebra & Number Theory.

[9] J. Migliore and U. Nagel, *A tour of the Weak and Strong Lefschetz properties*, preprint 2011.

[10] J. Migliore and F. Zanello, *The strength of the Weak Lefschetz Property*, Illinois J. Math. **52**, no. 4 (2008), 1417–1433.

[11] R. Stanley, *Weyl groups, the hard Lefschetz theorem, and the Sperner property*, SIAM J. Algebraic Discrete Methods **1** (1980), 168–184.

[12] J. Vallès, *Variétés de type Togliatti*, C. R. Acad. Sci. Paris, Ser. I **343** (2006), 411-414

[13] J. Watanabe, *The Dilworth number of Artinian rings and finite posets with rank function*, Commutative Algebra and Combinatorics, Advanced Studies in Pure Math. Vol. 11, Kinokuniya Co. North Holland, Amsterdam (1987), 303–312.