QUESTIONS ON LEFSCHETZ PROPERTIES AND h-VECTORS

SATOSHI MURAI

In this note, I summarize open problems which I will introduce in my talk. The first page is just an introduction of notation, so you can skip.

1. Preliminary

Stanley–Reisner rings. The main topic in my talk is Stanley–Reisner rings. Given a simplicial complex Δ on the vertex set V, its Stanley–Reisner ring $K[\Delta]$ is defined as follows: Let $S = K[x_v : v \in V]$ be a polynomial ring over a field K. For a subset $F = \{i_1, \dots, i_k\} \subset V$, we write $x_F = x_{i_1} x_{i_2} \cdots x_{i_k}$. Then the monomial ideal

$$I_\Delta = (x_F : F \subset V, F \notin \Delta)$$

is called the *Stanley–Reisner ideal* of Δ. The *Stanley–Reisner ring* $K[\Delta]$ of Δ is the quotient ring

$$K[\Delta] = S/I_\Delta.$$

h-vectors of Cohen–Macaulay complexes. For a simplicial complex Δ, let $f_i = f_i(\Delta) = \#\{F \in \Delta : \#F = i+1\}$. The *dimension* of Δ is $\max\{k : f_k(\Delta) \neq 0\}$. The vector

$$f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$$

is called the *f-vector* of Δ, where $f_{-1} = 1$ and where $d-1$ is the dimension of Δ. Then the *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ of Δ is defined by the relation

$$\sum_{i=0}^{d} h_i(\Delta) t^{d-i} = \sum_{i=0}^{d} f_{i-1}(t-1)^{d-i}.$$

The h-vector has the following algebraic description: Suppose that $K[\Delta]$ is Cohen–Macaulay and that the dimension of Δ is $d-1$. Then, assuming that K is infinite, there is a sequence of linear forms $\Theta = \theta_1, \dots, \theta_d$ such that $S/(I_\Delta + (\Theta))$ is Artinian. This sequence of linear forms is called a *linear system of parameters* (l.s.o.p. for short). In this setting, the h-vector is the Hilbert function of $S/(I_\Delta + (\Theta))$, namely

$$\dim_K \left(S/(I_\Delta + (\Theta))\right)_i = h_i(\Delta)$$

for $i = 0, 1, \dots, d$.

Lefschetz properties of Stanley–Reisner rings. Stanley–Reisner rings are not Artinian, so we define Lefschetz properties as follows: A simplicial complex Δ is said to have the WLP (resp. SLP) (over K) if it is Cohen–Macaulay and there is an l.s.o.p. Θ of $K[\Delta]$ such that $S/(I_\Delta + (\Theta))$ has the WLP (resp. SLP).

2. Conjectures and Problems

The most important problem on this topic is the following.

Problem 2.1. Find an application of Lefschetz properties to h-vectors.

Below I introduce some open problems. But they will be difficult since they are the problems which people could not solve. So, finding a NEW application is most important (and will be much easier than solving problems by other peoples).

2.1. **Polytopes and spheres.** The following is the long standing open conjecture.

Conjecture 2.2 (Algebraic g-conjecture)**.** Triangulations of a sphere (or more generally, Gorenstein Stanley–Reisner rings) have the SLP (or WLP).

At the moment, it seems to be hopeless to prove the above conjecture. On the other hand, it would be interesting to study special cases. I give one example. For a finite graded poset P of rank d, let $\Delta(P)$ be its order complex, namely,

$$\Delta(P) = \{\{\sigma_1, \cdots, \sigma_k\} \subset P : \sigma_1 < \cdots < \sigma_k\}.$$

If $\Delta(P)$ is Cohen–Macaulay then the linear forms $\theta_1, \ldots, \theta_d$, defined by $\theta_i = \sum_{\mathrm{rank}\sigma = i} x_\sigma$, is an l.s.o.p. of $K[\Delta(P)]$.

Problem 2.3. With the same notation as above, prove that if $\Delta(P)$ is Gorenstein then $S/(I_{\Delta(P)} + (\theta_1, \ldots, \theta_d))$ has the SLP (or WLP) in characteristic 0.

2.2. **Barycentric subdivisions.** For a simplicial complex Δ, its barycentric subdivision $\mathrm{sd}(\Delta)$ is defined to be the order complex of Δ (regarding Δ as a poset), namely,

$$\mathrm{sd}(\Delta) = \{\{F_1, F_2, \ldots, F_k\} \subset \Delta : \emptyset \neq F_1 \subset F_2 \subset \cdots \subset F_k\}.$$

Conjecture 2.4 (Kubitzke–Nevo [KN])**.** Let Δ be a Cohen–Macaulay simplicial complex of dimension $d-1$. Let Θ be a general l.s.o.p. of $K[\mathrm{sd}(\Delta)]$, then there is a linear form w such that the multiplication

$$\times w^{d-i-1} : (S/I_{\mathrm{sd}(\Delta)} + (\Theta))_i \longrightarrow (S/I_{\mathrm{sd}(\Delta)} + (\Theta))_{d-i-1}$$

is injective for all $i \leq \frac{d}{2}$.

The conjecture is known to be true for shellable simplicial complexes.

2.3. **Doubly Cohen–Macaulay simplicial complexes.** A simplicial complex Δ is said to be 2-*CM* if every vertex deletion $\Delta - v = \{F \in \Delta : v \notin F\}$ is CM, equivalently, $K[\Delta]$ is level whose socle degree is $\dim \Delta + 1$.

Conjecture 2.5 (Börner–Swartz [Sw])**.** Let Δ be a 2-CM simplicial complex of dimension $d-1$ and Θ a general l.s.o.p. of $K[\Delta]$. Then there is a linear form w such that the multiplication

$$\times w^{d-2i} : S/(I_\Delta + (\Theta))_i \to S/(I_\Delta + (\Theta))_{d-i}$$

is injective for $i \leq \frac{d}{2}$.

Conjecture 2.6 (Swartz [Sw])**.** If Δ is a 2-CM simplicial complex of dimension $d-1$, then the vector $(h_d - h_0, h_{d-1} - h_1, \ldots, h_{\lfloor \frac{d}{2} \rfloor - 1} - h_{\lfloor \frac{d}{2} \rfloor})$ is the sum of $h_d - 1$ M-vectors (O-sequences).

The above two conjectures are known to be true for Matroid complexes and, more generally, simplicial complexes which admit convex ear decompositions.

2.4. **Other problems.** The following problem comes from a question on triangulations of manifolds (I omit the backgrounds).

Problem 2.7. (solved) Let $R = S/I = R_0 \bigoplus R_1 \bigoplus R_2 \bigoplus R_3 \bigoplus R_4$ be an Artinian graded K-algebra such that

- R has the WLP
- $\dim_K R_2 = \dim_K R_3$
- there is a non-zero ideal $J \subset \mathrm{Soc}(R)_3 = (0 :_R \mathbf{m})_3$ such that R/J is Gorenstein

Does I have a generator of degree 4?

The solution of the above problem will give an affirmative answer to [NS, Problem 5.3]. *It was pointed out by Juan that there is a counterexample of Problem 2.7.* Thanks Juan!!

Problem 2.8. Let $R = \bigoplus_{i=0}^{c} R_i$ be an Artinian Gorenstein graded K-algebra which has the SLP. Let $I \subset R$ an ideal of R and $h_i = \dim_K (R/I)_i$ for $i = 0, 1, \ldots, c$. Then $h_i \geq h_{c-i}$ for $i \leq \frac{c}{2}$ since the multiplicaiton $\times w^{c-2k} : I_k \to I_{c-k}$ is injective for $k < \frac{c}{2}$, where w is a Lefschetz element. Is it true that $(h_0 - h_c, h_1 - h_{c-1}, \ldots, h_{\lfloor \frac{c}{2} \rfloor - 1} - h_{\lfloor \frac{c}{2} \rfloor})$ is an M-vector?

The problem is known to be true when R is a monomial complete intersection. The above problem is motivated from the following question: Let Δ be the boundary complex of a simplicial d-polytope, and let $\Gamma \subset \Delta$ be a subcomplex of Δ whose geometric realization is homeomorphic to a $(d-1)$-ball. Is the g-vector of the boundary of Γ an M-vector?

References

[KN] M. Kubitzke and E. Nevo, The Lefschetz property for barycentric subdivisions of shellable complexes, *Trans. Amer. Math. Soc.* **361** (2009), 6151–6163.

[NS] I. Novik and E. Swartz, Socles of Buchsbaum modules, complexes and posets, *Adv. Math.* **222** (2009), 2059–2084.

[Sw] E. Swartz, g-elements, finite buildings and higher Cohen-Macaulay connectivity, *J. Combin. Theory, Ser. A,* **113** (2006), 1305–1320.

Satoshi Murai, Department of Mathematical Science, Faculty of Science, Yamaguchi University, 1677-1 Yoshida, Yamaguchi 753-8512, Japan.