Problems of SLP and WLP (revised Aug. 18, 2012.)

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Junzo Watanabe

This is a list of problems most of which I asked myself and I have been unable to answer. Some problems should be answered immediately by specialists.

Problems on Strong Lefschetz property

1. Suppose that $A = \oplus_{i=1}^{c} A_i$ is a complete intersection, where $A_0 = K$ is a field, and $A = K[A_1]$.
   (a) Prove that A has the Sperner Property.
   (b) If characteristic of K is 0, prove that A has the SLP.
2. Suppose that $A = \oplus_{i=1}^{c} A_i$ is a Gorenstein ring, where $A_0 = K$ is a field of characteristic 0, and $A = K[A_1]$. Suppose that the symmetric group acts on A as the permutation of variables.
   (a) Prove that A has the Sperner Property.
   (b) If characteristic of K is 0, prove that A has the SLP.
3. Let $A = \oplus_{i=0}^{c} A_i = K[x_1, \cdots, x_n]/(x_1^{d_1}, \cdots, x_n^{d_n})$ is a monomial complete intersection. Put $L = x_1 + \cdots + x_n$, and let M_i be the matrix for the multiplication map $L^{c-2i} : A_i \to A_{c-i}$ on the bases of monomials. Compute the determinant of M_i.

   The determinant $\det M_i$ is known if either $n = 2$ or $d_1 = d_2 = \cdots = d_n = 2$.
4. (Assume that char $K = 0$.) In $K[x_1, \cdots, x_n]_d$, how is it possible to choose a set of n algebraically *dependent* elements

   $$f_1, f_2, \cdots, f_n$$

   such that any $n-1$ elements in it are algebraically independent. Say anything about this statement. For the time being it is enough to assume that $n = 3$.

   The necessity of this question arises from a desire to determine the forms in 6 variables with zero Hessian. See the following problem.

   Assume that three elements $f, g, h \in K[x_1, x_2]_d$ are linearly independent. (Automatically they are algebraically dependent.) Introduce three indeterminate x_3, x_4, x_5 and let $F = fx_3 + gx_4 + hx_5$. Then the Hessian determinant F vanishes but no variables can be eliminated from F by means of a linear transformation of the variables. In other words $\frac{\partial F}{\partial x_1}, \cdots, \frac{\partial F}{\partial x_5}$ are linearly independent. Moreover any polynomial in $K[x_1, x_2][F]$ has zero Hessian. These exhaust homogeneous polynomials in five variables with zero Hessian which involve 5 variables properly(Gordan-Noether). Bearing this in mind go to the next problem.

5. Determine the homogeneous polynomials whose Hessian identically vanishes in

$$K[x_1, \cdots, x_6].$$

Suppose that $F = F(x_1, \cdots, x_6)$ is a form with zero Hessian. We say that F reduces (to a form in less than 6 variable) if F mod (a linear form) is a form with zero Hessian.

Choose three elements $(f, g, h) \subset K[x_1, x_2, x_3]_d$ that are linearly independent. Assume that f, g, h are algebraically dependent. E.g. $f = x_1^4, g = x_1^2 x_2 x_3, h = x_2^2 x_3^2$. Assume that

$$\left(\frac{\partial(f, g, h)}{\partial(x_1, x_2, x_3)} \right) \tag{1}$$

has rank 2. Introduce three variables x_4, x_5, x_6 and let $F = fx_4 + gx_5 + hx_6$. Then any polynomial in $K[x_1, x_2, x_3][F]$ has zero Hessian. (This is a fact easily verified.)

I conjecture simple-mindedly that these exhaust all homogeneous polynomial $F' \in K[x_1, \cdots, x_6]$ which has zero Hessian which does not reduce to a five variable form with zero Hessian.

Moreover I conjecture that F reduces to a five variable case if and only if

$$\text{corank} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) > 1$$

Problems on weak Lefschetz property

6. Assume that char $K = 0$. Let $K[x_1, \cdots, x_n]$ be the polynomial ring. Let $I \subset R$ be a homogeneous ideal. Let m be the homogeneous maximal ideal. Define the ideals I_i for $i = 0, 1, \cdots$ inductively as follows: $I_0 = I, I_i = mI_{i-1} : y$, where y is a general element of R. Let $J = \cup I_i$. Prove that

$$\mu(J) = \text{length} R/(mI + (y)).$$

(We have not found a counter example in characteristic $K = 0$ and $n \leq 4$.) If $n = 5$, Murai gave a counter example. So this is a problem only for $n \leq 4$. Once this is proved then it implies that an ideal I has the Rees property if and only if I is m-full in the polynomial ring.

In 1987, I discussed the problem of Rees for $n = 3$, which asks for what ideal I is it true that $\mu(I) \geq \mu(J)$ for all ideals J such that $J \supset I$ with Craig Huneke. We tried to prove that I has this property if and only if I is m-full in three variable case. We were unable to prove or disprove it. Huneke's comment: This is too good to be true.

(This part was added later:) Murai found a counter example to this "conjeture" even in $n = 3$. So it seems that there are many Artinian algebras with (1) Unimodal Hilbert function and (2) Sperner property but without WLP.)

Problems of commutative rings

7. Let A be an Artinian local ring. Then it holds that

$$\mu(m^j) \leq d(A) \leq \text{length}(A/yA),$$

where j is a non-negative integer and y is a general element. For which A does it fails to have the the second equality? All people seem to have been too busy with the equality

$$\mu(m^j) = \text{length}(A/yA)$$

and there are not many papers on the second equality.

8. Let $R = K[x_1, \cdots, x_n]$ and $m = (x_1, \cdots, x_n)$. Suppose that R/I does not have the WLP. Let $J = mI$. Does R/J have the WLP?

   The reason I am interested in this problem is this: I conjecture that $d(R/mI) =$ length $R/mI + lR$, where I is an arbitrary (homogeneous) maximal primary ideal and l is a general linear element. If I am to find a counter-example to this conjecture, a good candidate is I such that R/I does not have the WLP.

9. What is the module M of finite length with

$$\tau(M) = \mu(M).$$

10. Suppose that (x_{ij}) is the $r \times s$ generic matrix. Let $R = K[\{x_{ij}\}]$ be the polynomial ring. Let M be the cokernel of the map $R^s \to R^r$ defined by the homomorphism (x_{ij}). When does this have a symmetric minimal free resolution? (This should be known.) Suppose this is the case. Let $\overline{M}$ be the reduction of M by a regular sequence consisting of linear forms. Then we should have

$$\tau(\overline{M}) = \mu(\overline{M}).$$

   Does $\overline{M}$ have the SLP?

11. Suppose that K is a field of characteristic zero. Suppose that K contains enough transcendental elements over . Let $R = K[x_1, \cdots, x_n]$ be the polynomial ring. (It is possible to define a generic complete intersection.) Suppose that R/I is a generic complete intersection. How can we conclude that R/I has the SLP?

Problems of polynomial rings over a field of characteristic zero

12. What is a good reference for the following?

   Let $R = K[x_1, \cdots, x_n]$ be the polynomial ring over K. Suppose that $f_1, \cdots, f_n$ be a sequence of homogeneous polynomials (say of the same degree).

   Let r be the transcendence degree of the function field $K(f_1, \cdots, f_n)$. Then r equals the rank of the matrix

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right) \tag{2}$$

13. Suppose that M is a graded module of finite colength over a $K[x_1, \cdots, x_n]$. Assume that $Ext^n_R(M, R)(-n) \cong M$. Under what condition can we prove that M has the SLP? For general n this is too difficult. Can we say anything if $n = 2$? This helps us understand the SLP of Gorenstein algebras in embedding codimension three.

    Can we say anything about such M, if M is generated by two elements.

14. Suppose that A is a standard Artinian Gorenstein algebra. Generally speaking $A/(0 : l)$ is a Gorenstein algebra for a linear form l (or any non-zero element l). Does this have any meaning if we consider A as the cohomology ring of an algebraic variety?

15. Suppose that $A = K[x_1, \cdots, x_n]/(x_1^d, \cdots, x_n^d)$. Let G be a subgroup in Σ_n generated by reflections. Assume that G leaves the element $x_1 + \cdots + x_n$ invariant. Then A^G has a complete intersection with the SLP with a Lefschetz element l. Are there some cases where A^G have combinatorial meaning? For example if $G = \Sigma_n$, then A^G may be interpreted as the lattice of Young diagrams contained in a rectangle.

16. Consider the algebra

    $$A = K[x_1, \cdots, x_n]/(x_1^d - x_2^d, , x_2^d - x_3^d, \cdots, x_{n-1}^d - x_n^d, x_1^m + \cdots + x_n^m).$$

    For which (n, d, m), is A an Artinian ring? Suppose it is Artinian. Decompose the algebra A into irreducible modules as a module of Σ_n, where Σ_n is the symmetric group acting on A by permutation of the variables. We have done this for $m = d$.

17. Let $R = K[x_1, x_2, x_3]$ be the polynomial ring. Put

    $$p_d = x_1^d + x_2^d + x_3^d.$$

    For what choice of degrees i, j, k, is the algebra $A = R/(p_i, p_j, p_k)$ is a complete intersection. Conjecture is this: Suppose that $\mathrm{GCD}(i, j, k) = 1$ without loss of generality. Then A is a complete intersection if and only if $ijk \equiv 0 \bmod 6$. The same question can be asked for complete symmetric polynomials for power sum symmetric functions.