

# Problem List for the Workshop on Lefschetz Properties

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BY WAY OF EXPLANATION for what might appear as a set of strange problems: my training as a mathematician was in algebraic topology in which I worked for something like 25 years before being attracted to invariant theory ,..., which brings me to a WorkShop on Lefschetz properties. A subject to which I am a newcomer. The problems that follow are needless to say strongly influenced by this background, where characteristic  $p \neq 0$  plays an important role.

## Problems about Characteristic $p \neq 0$

**BACKGROUND FOR PROBLEM 1:** One of fundamental differences between algebra in characteristic  $p \neq 0$  as opposed to over fields of characteristic zero such as  $\mathbb{Q}, \mathbb{R}$ , or  $\mathbb{C}$  is the **Frobenius homomorphism**,  $\lambda$ , which raises an element  $a$  to its  $p$ -th power  $\lambda(a) = a^p$  and respects the linear structure. It leads to a rich and interesting theory reaching a high point in the Frobenius functor  $\mathbf{F}$  of C. Peskine and L. Szpiro [18]. The homomorphism  $\lambda$  has lots of nice properties, e.g., in a polynomial algebra  $S = \mathbb{F}[z_1, \dots, z_n]$  it preserves maximal primary ideals;<sup>1</sup> if  $I \subset \mathbb{F}[z_1, \dots, z_n]$  is a maximal primary irreducible ideal then so is the ideal  $I^{[p]}$  generated by the  $p$ -th powers of elements of  $I$  [19] (see [16] or [14] §II.6). So the graded analog<sup>2</sup> of the Frobenius functor  $\mathbf{F}$  will preserve Poincaré duality quotient algebras of  $S$ . However  $\mathbf{F}$  seems not to preserve the strong Lefschetz property if  $n > 1$ .

**PROBLEM 1:** Let  $I \subseteq S = \mathbb{F}[z_1, \dots, z_n]$  be a maximal primary ideal such that  $S/I$  has the weak Lefschetz property. Does  $S/I^{[p]}$  have the weak Lefschetz property?

**BACKGROUND FOR PROBLEM 2:** Let  $\mathbb{F}_q$  be the finite field with  $q = p^v$  elements where  $p \in \mathbb{N}$  is a prime integer. Denote by  $S = \mathbb{F}_q[V]$  the algebra of polynomial functions on  $V = \mathbb{F}_q^n$  and by  $z_1, \dots, z_n$  a basis for the linear forms  $V^*$ . If  $I \subseteq S = \mathbb{F}_q[z_1, \dots, z_n]$  is a maximal primary irreducible ideal such that  $S/I$  has the strong Lefschetz property then  $A = S/I^{[p]}$  might not have the strong Lefschetz property if  $n > 1$ , but, if we form the quotient algebra  $B = A/\text{Ann}_A((z_1 \cdot z_2 \cdots z_n)^{q-1})$  then, at least, it contains a linear element  $\ell$  all of whose powers  $\ell^i$  for  $i$  up to the top non zero degree of  $A$  are nonzero.<sup>3</sup> The algebra  $B$  just described is what a topologist might refer to as the **dual** of  $(z_1 \cdot z_2 \cdots z_n)^{q-1}$  in  $A$  (see e.g., [26] §5 for why).

**PROBLEM 2:** With the previous notations and assumptions does the algebra  $B$  dual to  $(z_1 \cdot z_2 \cdots z_n)^{q-1}$  in  $A$  have the strong Lefschetz property?

Proposition 3.9 of [6] seems relevant to this problem. An added complication though, even if the answer were yes, is to find a direct description of a set of generators for the kernel  $J$  of the map  $S \longrightarrow B$  in terms of generators for  $I$ ; or what would be perhaps nicer, a *functor* that takes  $I$  to  $J$  directly. If we knew a Macaulay dual (or apolar form, see the discussion of Macaulay's Inverse Systems to follow)  $\theta$  for  $I$  then  $\theta^q$  would be a Macaulay dual for  $J$ ; this still leaves open the question about generators.

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<sup>1</sup>These are the ideals defining Gorenstein–Artin quotients of  $S$ .

<sup>2</sup>The construction of such a graded analog was shown to me by Nicole Nossem about ten years ago.

<sup>3</sup>Certainly in an algebra with the Strong Lefschetz property there must be such an element.

## Problems about Invariant Theory

For unexplained notations please see [20].

**BACKGROUND FOR PROBLEM 3:** A useful paradigm in representation theory (and elsewhere) is to think of the symmetric group  $\Sigma_n$ , i.e., the permutations of the set  $[n] = \{1, 2, \dots, n\}$ , as the general linear group of the *field with one element*: a set being regarded as a vector space over this field. One then has a basis for seeking analogies. The invariant theory of the symmetric group is fairly well worked out, and an analog of the Fundamental Theorem on Symmetric Functions for the general linear group  $\text{GL}(n, \mathbb{F}_q)$  of the Galois field  $\mathbb{F}_q$  with  $q = p^\nu$  elements,  $p \in \mathbb{N}$ , was discovered by L.E. Dickson [4]. In modern language, if  $V = \mathbb{F}_q^n$  is the  $n$ -dimensional vector space over the finite field  $\mathbb{F}_q$  and we define

$$\Phi(X) = \prod_{v \in V^*} (X - v) \in \mathbb{F}_q[z_1, \dots, z_n][X]$$

then  $\Phi(X)$  is a  $q$ -polynomial in the sense of [17], so when written as a polynomial in  $X$  the coefficients of  $X^i$  all vanish except those where  $i$  is a power of  $q$ . The coefficients are clearly invariant under the action of  $\text{GL}(n, \mathbb{F}_q)$  on  $\mathbb{F}_q[z_1, \dots, z_n]$ . So one can write<sup>4</sup>

$$\Phi(X) = \sum_{i=0}^{n-1} \mathbf{d}_{n,i} X^{q^i} \in \mathbb{F}_q[z_1, \dots, z_n]^{\text{GL}(n, \mathbb{F}_q)}[X]$$

defining the **Dickson polynomials**,<sup>5</sup>  $\mathbf{d}_{n,i} \in \mathbb{F}_q[z_1, \dots, z_n]^{\text{GL}(n, \mathbb{F}_q)}$  of degrees  $q^n - q^i$ , for  $i = 0, 1, \dots, n-1$ . What Dickson showed is that the algebra of invariants  $\mathbf{D}(n) = \mathbb{F}_q[z_1, \dots, z_n]^{\text{GL}(n, \mathbb{F}_q)}$  is a polynomial algebra generated by  $\mathbf{d}_{n,0}, \mathbf{d}_{n,1}, \dots, \mathbf{d}_{n,n-1}$ . So  $\mathbb{F}_q[z_1, \dots, z_n]/(\mathbf{d}_{n,0}, \dots, \mathbf{d}_{n,n-1}) = \mathbb{F}[z_1, \dots, z_n]^{\text{GL}(n, \mathbb{F}_q)}$  is a complete intersection algebra. This is analogous to the Fundamental Theorem on Symmetric Polynomials which says that  $\mathbb{F}[x_1, \dots, x_n]^{\Sigma_n}$  is a polynomial algebra generated by the elementary symmetric polynomials  $e_1, \dots, e_n \in \mathbb{F}[x_1, \dots, x_n]$  etc.

**PROBLEM 3:** *The Dickson coinvariant algebra  $\mathbb{F}_q[z_1, \dots, z_n]^{\text{GL}(n, \mathbb{F}_q)}$  is a complete intersection algebra. What Lefschetz properties does it have? How do they depend on the number of field elements and the number of variables?*

**BACKGROUND FOR PROBLEM 4:** In the same paper [27] of R. Steinberg discussed further on under *Problems from Macaulay's Inverse System* Steinberg proves that if the ring of coinvariants  $\mathbb{C}[V]_G$  of a finite subgroup  $G \leq \text{GL}(n, \mathbb{C})$  is a Poincaré duality algebra then  $\mathbb{C}[V]^G$  is a polynomial algebra, so  $\mathbb{C}[V]_G$  was in fact a complete intersection. Here  $\mathbb{C}$  denotes the complex number field. Extending this to the nonmodular case (the case where the order  $|G|$  of  $G$  is invertible in the field  $\mathbb{F}$ ) is not completely trivial, and was first done for finite fields by T.-C. Lin in her Göttingen Doktorarbeit [11]. As a purely(?) invariant theory question one has been confronted with the following problem for close to half a century.

**PROBLEM 4:** *Are there, and if so what are they, analogs of Steinberg's theorems in the modular case, i.e., the case where  $G \leq \text{GL}(n, \mathbb{F})$  and the characteristic of  $\mathbb{F}$  divides the order  $|G|$  of  $G$ ? Specifically, if  $\mathbb{F}[V]_G$  is a Poincaré duality algebra must it be a complete intersection?*

To paraphrase [6] the Lefschetz properties provide a new point of view and new tools to study (co)invariant algebras. So perhaps they are of use in connection with this problem.

<sup>4</sup>The reason for the indexing is explained in [20] Chapter 6 where a proof of Dickson's Theorem can also be found.

<sup>5</sup>Warning! In the theory of finite fields there are *other* polynomials referred to as Dickson polynomials: Namely the polynomials in one variable over the field that as a function of the field elements act as a permutation.

The papers [5], [11], [22], [23], [24], and [25] are all contributions to solving this problem and contain further references. It can happen that  $\mathbb{F}[V]_G$  is a complete intersection *without*  $\mathbb{F}[V]^G$  being a polynomial algebra. In other words the **Hilbert ideal**  $\mathfrak{h}(G) \subset \mathbb{F}[V]$ , i.e., the ideal generated by the invariant forms of strictly positive degree, might be generated by a regular sequence, but yet  $\mathbb{F}[V]^G$  need not a polynomial algebra; e.g., the Hilbert ideal of the alternating subgroup  $A_n$  of  $\Sigma_n$  has this property in characteristic  $p$  if  $p \leq n$ .

## Problems about Generalized Invariants

The study of the modular invariant theory of reflection groups led to the introduction of a very interesting mechanism for construction complete intersection algebras as quotients of  $\mathbb{F}[z_1, \dots, z_n]$ . Briefly, this runs as follows (see e.g., [3], [7], or [15]).

An element  $s \in \mathrm{GL}(n, \mathbb{F})$  is a **reflection** if it has finite order  $|s|$  and  $\mathrm{Im}(s - 1) \subseteq V$  is a 1-dimensional subspace of  $V$  called the **root space** of  $s$  and denoted by  $R_s$ . A nonzero vector  $x_s \in \mathrm{Im}(s - 1)$  is called a **root vector** of  $s$  and  $\ker(s - 1)$  the **reflecting hyperplane** of  $s$  and is denoted by  $H_s$ . The reflecting hyperplane of  $s$  is a codimension one subspace of  $V$  left pointwise fixed by  $s$ . Hence  $s$  has 1 as an eigenvalue of multiplicity  $n - 1$  and its characteristic polynomial  $p(X)$  splits into  $(X - 1)^{n-1}(X - \det(s))$ . Note that of the three quantities:

- a reflection  $s \in \mathrm{GL}(n, \mathbb{F})$ ,
- a root vector  $x_s \in V$  for the reflection  $s \in \mathrm{GL}(n, \mathbb{F})$ ,
- an axis  $\ell_s \in V^*$  for the reflection  $s \in \mathrm{GL}(n, \mathbb{F})$ ,

any two determine the third uniquely because of the basic equation

$$s(v) = v - \ell_s(v) \cdot x_s \quad \forall v \in V.$$

relating them. If  $s \in \mathrm{GL}(n, \mathbb{F})$  is a reflection we may write

$$s(f) = f + \ell_s \cdot \Delta_s(f) \quad \forall f \in \mathbb{F}[V]$$

defining the operator  $\Delta_s : \mathbb{F}[V] \rightarrow \mathbb{F}[V]$  lowering degrees by one.

Given  $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ , a nonempty set of reflections, we define<sup>6</sup>

$$I(\mathcal{S}) = \left\{ f \in \overline{\mathbb{F}[V]} \mid \Delta_{s_1} \cdots \Delta_{s_{\deg(f)}}(f) = 0 \quad \forall s_1, \dots, s_{\deg(f)} \in \mathcal{S} \right\}.$$

Then  $I(\mathcal{S}) \subset \mathbb{F}[V]$  is a (proper) ideal, an element  $f \in \mathbb{F}[V]$  belongs to  $I(\mathcal{S})$  if and only if  $\Delta_s(f) \in I(\mathcal{S})$  for all  $s \in \mathcal{S}$ , so  $I(\mathcal{S})$  is closed under all the operations  $\Delta_s$  for  $s \in \mathcal{S}$ , and if  $\mathcal{I}$  denotes the set of  $\mathbb{F}$ -subspaces of  $\mathbb{F}[V]$  which are closed under all the operations  $\Delta_s$  for  $s \in \mathcal{S}$  then  $I(\mathcal{S})$  is the unique maximal element in  $\mathcal{I}$ . A result of V. Kac and D. Peterson [7] shows that the ideal  $I(\mathcal{S})$  is generated by a regular sequence<sup>7</sup> of length  $n$  so the quotient algebra  $\mathbb{F}[V]/I(\mathcal{S})$  is a standard graded complete intersection algebra.

**PROBLEM 5:** If  $\mathcal{S} \subset \mathrm{GL}(n, \mathbb{F})$ , is a nonempty set of reflections what Lefschetz properties, if any, does the quotient algebra  $\mathbb{F}[V]/I(\mathcal{S})$  have?

This seems interesting even in characteristic zero, e.g., for  $\mathbb{F} = \mathbb{C}$ , since these examples of complete intersections include the coinvariant algebras of the finite complex reflection groups.

<sup>6</sup> Here  $\overline{\mathbb{F}[V]}$  denotes the augmentation ideal of  $\mathbb{F}[V]$ .

<sup>7</sup> See also [15] §2; the essential point being a result of W. Vasconcelos [28].

## Problems about Macaulay's Theory of Inverse Systems<sup>8</sup>

For unexplained notations please see [14].

**BACKGROUND FOR PROBLEM 6:** I will make use of Macaulay's theory of *modular systems* (see [12] and [13]) in the language of *apolarity* or *inverse systems*.<sup>9</sup> So,  $\mathbb{F}[z_1, \dots, z_n]$  denotes a polynomial algebra over the field  $\mathbb{F}$  with the standard grading and  $\mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$  a second such algebra, but graded so that  $\deg(z_i^{-1}) = -1$  for  $i = 1, \dots, n$ . The monomial  $z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$  of  $\mathbb{F}[z_1, \dots, z_n]$  is denoted by  $z^E$  where  $E = (e_1, e_2, \dots, e_n) \in \mathbb{N}_0^n$  is an index sequence, and similarly  $z^{-F} = z_1^{-f_1} z_2^{-f_2} \cdots z_n^{-f_n}$  for an inverse monomial in  $\mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$ , with  $F = (f_1, \dots, f_n) \in \mathbb{N}_0^n$ . The **contraction pairing**

$$z^E \cap z^{-F} = \begin{cases} z^{-(E-F)} & \text{if } E - F \in \mathbb{N}_0^n, \text{ or} \\ 0 & \text{otherwise,} \end{cases}$$

makes  $\mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$  into a module over  $\mathbb{F}[z_1, \dots, z_n]$  and F.S. Macaulay showed that that there is a bijective correspondence between cyclic submodules of the inverse polynomial algebra  $\mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$  and maximal primary irreducible ideals of  $S = \mathbb{F}[z_1, \dots, z_n]$  obtained by taking annihilators with respect to the contraction pairing. As nice as this theory is it leaves the following very basic problem almost untouched.

**PROBLEM 6:** *Under what conditions on the inverse form  $\theta \in \mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$  will the corresponding ideal  $I(\theta) \subset \mathbb{F}[z_1, \dots, z_n]$  be a complete intersection ideal?*

Contrast this problem with the construction using  $\Delta$ -operators from the preceding section which a priori yields complete intersections. For the Poincaré duality quotients of  $\mathbb{F}[z_1, \dots, z_n]$  one has in any case the additional problem.

**PROBLEM 7:** *What conditions on the inverse form  $\theta \in \mathbb{F}[z_1^{-1}, \dots, z_n^{-1}]$  guarantee that the corresponding Poincaré duality algebra  $\mathbb{F}[z_1, \dots, z_n]/I(\theta)$  has one of the Lefschetz properties?*

**BACKGROUND FOR PROBLEM 8:** In the paper [27] of R. Steinberg he shows how to construct for a complex reflection group an additive basis for the ring of coinvariants  $\mathbb{C}[V]_G$  as the divisors of a basic harmonic form. For example, if  $G$  is the symmetric group acting in its tautological representation on the set  $X = \{x_1, \dots, x_n\}$  this basis consists of the divisors of the discriminant  $\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$ . To do this Steinberg essentially constructs<sup>10</sup> an apolar form for the Hilbert ideal  $\mathfrak{h}(G)$  of  $G$ . For some interesting examples in this connection see [8]. Steinberg's result<sup>11</sup> has been extended to some modular cases in [2].

**PROBLEM 8:** *What is the analog of Steinberg's result in the modular case? In other words, assume  $G \leq \text{GL}(n, \mathbb{F})$  has polynomial invariants and construct an apolar form for the Hilbert ideal, i.e., the ideal generated by the invariant forms of strictly positive degree. Relate that form to the configuration of hyperplanes associated to the group and use it to construct (if possible) an  $\mathbb{F}$ -basis for the coinvariant algebra  $\mathbb{F}[V]_G$  and determine what Lefschetz properties that algebra has.*

<sup>8</sup> Also called Macaulay Duals, or Apolar Forms

<sup>9</sup> Other formulations of this theory are possible; the most general known to me being in terms of local cohomology and the local Duality Theorem for Gorenstein algebras. This is discussed briefly in [25].

<sup>10</sup> To do so he makes use of an existence theorem for meromorphic solutions to partial differential equations with analytic coefficients: This is the partial differential equation referred to in the title of the paper. Is this *analysis* equivalent to Macaulay's *algebraic* theory of inverse / apolar elements in this context?

<sup>11</sup> It is implicitly mentioned in [6] §4.2 but I do not see where it was ever used in [6]: Was it? Does such a basis aide in verifying one of the Lefschetz properties for the coinvariant algebra? Some of the combinatorial techniques in [2] look promising in this direction.

See e.g., [6] Proposition 8.18, Problem 8.19, Remark 8.20, and Conjecture 8.24. The paper [25] contains a modest attempt to address these problems.

## Problems about Jordan Forms

**BACKGROUND FOR PROBLEM 9:** In [21] §19.1 it is argued that the similarity problem for matrices cannot have an *algebraic* solution. Roughly the argument goes as follows. Two  $n \times n$  matrices  $\mathbf{A}, \mathbf{B} \in \text{Mat}_{n,n}$  are similar if there is an invertible matrix  $\mathbf{P} \in \text{Mat}_{n,n}$  such that

$$(*) \quad \mathbf{B} = \mathbf{P} \cdot \mathbf{A} \cdot \mathbf{P}^{-1}.$$

By an algebraic solution to the problem of Jordan forms one might mean the following.

**SIMILARITY PROBLEM:** For each positive integer  $n$  find  $n^2$  polynomial functions<sup>12</sup>

$$f_{i,j} : \text{Mat}_{n,n} \longrightarrow \mathbb{C}, \quad i, j = 1, \dots, n,$$

such that for any matrix  $\mathbf{A} \in \text{Mat}_{n,n}$  the matrix  $(f_{i,j}(\mathbf{A}))$  is the Jordan canonical form of  $\mathbf{A}$ .

For this to work one would want that such functions  $f_{i,j}$  satisfy  $f_{i,j}(a_{r,s}) = f_{i,j}(b_{r,s})$  whenever the matrices  $\mathbf{A} = (a_{r,s})$  and  $\mathbf{B} = (b_{r,s})$  are related by an equation (\*). A moments thought shows one can reformulate this in the following invariant theoretic way: Find finitely many polynomial functions  $h_{i,j} \in \mathbb{C}[\text{Mat}_{n,n}]^{\text{GL}(n,\mathbb{C})}$ ,  $i \in \mathcal{I}, j \in \mathcal{J}$ , such that two matrices  $\mathbf{A}, \mathbf{B}$  have the same Jordan form if and only if  $h_{i,j}(a_{r,s}) = h_{i,j}(b_{r,s})$  for all  $i \in \mathcal{I}, j \in \mathcal{J}$ , and  $1 \leq r, s \leq n$ . This amounts to saying that the functions  $h_{i,j}$  separate the orbits of  $\text{Mat}_{n,n}$  under the conjugation action of  $\text{GL}(n, \mathbb{C})$ . This is not possible (see e.g., [21] Chapter 19), yet the method of [9] provides a quasi-algebraic way to construct Jordan normal form projections, i.e., projections onto the invariant subspaces where the matrix is represented by a single Jordan block, and therefore provides the sizes of the blocks. What follows is sort of a philosophical problem (perhaps only for me).

**PROBLEM 9:** Explain *why* the  $\mathfrak{sl}_2$  method is therefore so successfull despite this in studying the Jordan structure of nilpotent matrices arising from multiplication by a linear element  $\ell$  in a graded Artin algebra.

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<sup>12</sup>Other choices might lead to interesting results: The invariant theory of at least finite reflection groups is basically the same for polynomial functions and analytic functions. For  $\mathcal{C}^k$ -functions things get a bit dicey, and one can loose some differentiability. See [1] and the referneces there.

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